Generalizations of Kaprekar's transformations and Kaprekar-style transformations

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Abstract. In this paper we develop the research we started in [6, 7, 9, 10] on classical n-th Kaprekar's transformations, which are some functions defined on n-digit numbers (definition is given in the first section). This time we present and discuss some generalizations of classical Kaprekar's transformations which we call symmetric, permutational and general Kaprekar's transformations. We are especially interested in orbits of such transformations, namely we provide their full description for small n and also prove some general properties. We also mention some results of the attempt to introduce Kaprekar-style transformations on symmetric groups. Moreover, inspired by numerical calculations, we pose 4 conjectures.

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1. Introduction

In [5–12, 16, 17] the classical n-th Kaprekar's transformations were discussed, namely the transformations

$$T_n\colon \{0\} \cup \{\alpha\colon 10^{n-1} \leqslant \alpha < 10^n\} \to \{0\} \cup \{\alpha\colon 10^{n-1} \leqslant \alpha < 10^n\},$$

given by the formula

$$T_n(\alpha) := \sum_{k=1}^{n} (a_k - a_{n-k+1}) 10^{k-1} = a_n a_{n-1} \ldots a_1 - a_1 a_2 \ldots a_n,$$

for every $\alpha, n \in \mathbb{N},\ 10^{n-1} \leqslant \alpha < 10^n$, where

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$$0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant 9$$

denote all digits in decimal expansion of α and $T_n(0) = 0$.

In this paper we discuss orbits of some Kaprekar-style transformations. Note that a transformation F on a finite set X is a permutation (bijection) if and only if for every $x \in X$ there exists $n = n(x)$ such that $F^n(x) = x$, where F^n denotes the n-th iteration of F (that is, the n-fold composition). So to establish the nature of a permutation one can examine its orbits, that is sets of the form $\{F^n(x),\ n \in \mathbb{N}\}$ for every $x \in X$ (i.e. the sets of all iterations of F on a fixed x).

But the above is not true in general – if we consider an arbitrary transformation F of a finite set X then the set of all iterations on every $x \in X$ for each element can be presented graphically in Figure 1(a) below. Therefore instead of examining all iterations of F for every $x \in X$ one can examine only those which can be viewed at Figure 1(a) as a cycle, that is sets of the form $\{x_0 = F^{\nu+1}(x_0),\ F(x_0), \ldots F^{\nu}(x_0))\}$. Now each such defined set is called an **orbit of the transformation** F (in [9] it was called a minimal orbit). Moreover, if an orbit contains one element only, that is we have $F(x_0) = x_0$ then x_0 is called a **fixed point of the transformation** F. Note that orbits defined in such a way inherit the key property of orbits of permutations, namely every two orbits either are disjoint or coincide. Some exemplary sets of iterations and orbits of a transformation on a finite set are presented graphically in Figure 1(b) below.[1]

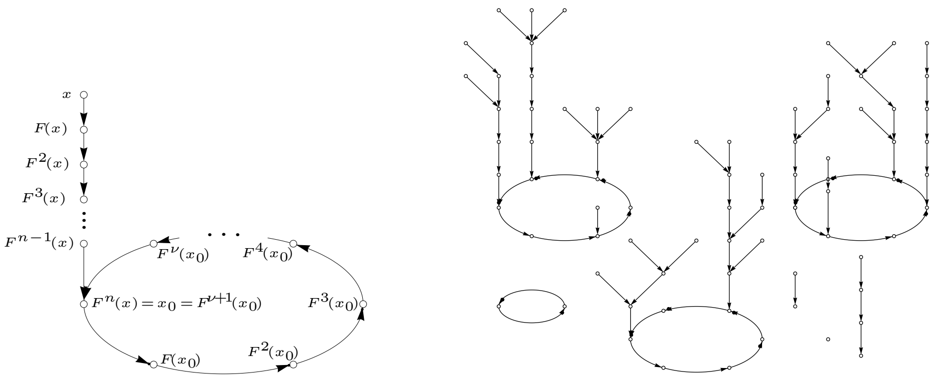

Fig. 1. (a) An example of iterations of F on x; (b) An example of F acting on some finite set

The paper is organized as follows. In Section 2 we examine orbits of n-th symmetric Kaprekar's transformations M_n. Their number and nature depends heavily on a parity of n, but not only. For instance, M_{13}-orbits have significantly different nature and there are much more fixed points of M_{15} than of any other M_n with $n \leqslant 20$. In Section 3 we define a family of permutational transformations and we examine orbits of one of them, denoted by Q_n. Most of Q_n-orbits have a surprising property – the sum of digits of each nontrivial element in every orbit is constant for a fixed n (except

[1] A very famous problem, usually referred to as (3x+1)-problem, Collatz problem or Syracuse problem, is also connected with examining orbits – this is the question whether the considered transformation has only one trivial orbit, see e.g. [14] for further details.

for Q_{3n} and Q_7). The examination of Q_n for several initial values of n allowed us, among the others, to discover (and prove) the form of some orbits for 5 infinite chains of Q_n. In Section 4 we define the family of general Kaprekar's transformations and discuss one of them denoted by D_n. Note that for $n = 2^k,\ k \in \mathbb{N}$ these transformations have only the trivial orbit, that is $\{0\}$, which takes place also for so-called Ducci's transformations. The structure of D_n-orbits in general is simpler than for T_n but their cardinalities increase rapidly with n. This behavior is different from the one of T_n as all T_n-orbits for $n \leqslant 50$ posses at most 7 elements. Finally in Section 5 we make an attempt to introduce Kapkrekar-style transformation in some algebraic structures. As the example we define two transformations in the spirit of Kaprekar on symmetric groups S_n. We also pose 4 conjectures.

2. Symmetric Kaprekar's transformations

In this section we consider a transformation that acts on nonnegative integers having **at most** n digits. However, since the value for each number (except for 0) is an n-digit number, and because of formula (1) below the transformations in question actually generalize the idea of T_n.

So, similarly as for T_n, let $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant 9$ be the sequence consisting of all digits of α in the decimal expansion of $\alpha < 10^n$ (filled with zeros at the beginning if necessary). For example, for $n = 5$ and $\alpha = 6103$, we have $a_1 = a_2 = 0, a_3 = 1, a_4 = 3$ and $a_5 = 6$.

Definition 2.1. *Let n be any natural number and for every $\alpha < 10^n$ let $a_1, .., a_n$ be a nondecreasing sequence of digits in the decimal expansion of length n of α. The n**-th symmetric Kaprekar's transformation** M_n is a function defined by*

$$M_n(\alpha) = \sum_{k=1}^{n} |a_k - a_{n-k+1}| 10^{k-1}. \tag{1}$$

Similarly as for classical Kaprekar's transformations T_n we have

$$M_n(a_1 \ldots a_n) = M_n \left(a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}\right)$$

for every permutation σ of $\{1, ..., n\}$.

Thus for example, $M_5(6103) = M_5(00136) = 63036$. Moreover, we have $M_6(6103) = M_6(000136) = 631136$ and $M_7(6103) = M_7(0000136) = 6310136$.

Note that values of M_n have a nice form, namely for every n we have

$$M_{2n+1}(\alpha) = b_1 b_2 \ldots b_n 0 b_n \ldots b_2 b_1,$$

and

$$M_{2n}(\alpha) = b_1 b_2 \ldots b_n b_n \ldots b_2 b_1,$$

where

$$0 \leqslant b_n \leqslant \ldots \leqslant b_2 \leqslant b_1 \leqslant 9.$$

Thus nontrivial M_n-orbits consists only of elements of one of the two forms defined above (in particular, they are n-digits numbers for $\alpha \neq 0$).

We shall give the full description of orbits for small n. Also we shall make some general observations and pose 2 conjectures.

2.1. M_n-orbits for odd n

We start our investigations with the full description of orbits for every odd $n \leqslant 17$. As noted above, the only elements that can appear in M_{2k+1}-orbits are of the form

$$b_1 b_2 \dots b_k 0 b_k \dots b_2 b_1, \qquad 0 \leqslant b_k \leqslant \dots \leqslant b_2 \leqslant b_1 \leqslant 9 \tag{2}$$

Fact 2.2. $M_3(a0a) = a0a$ *for* $a \in \{0, 1, \dots, 9\}$. *That means that each of ten M_3-orbits consist of 1 element only, i.e. a fixed point of M_3.*

Proof. $a0a$ are the only 3-digit numbers of the form (2). □

Fact 2.3. *M_5-orbits are the following:*
1-element orbits (fixed points): $\{0\}$, $\{21012\}$, $\{42024\}$, $\{63036\}$, $\{84048\}$
2-element orbits: 25 sets of the form

$$\{ab0ba,\ a(a-b)0(a-b)a\},$$

where $0 \leqslant b \leqslant a \leqslant 9$ *and* $ab0ba$ *is not a fixed point.*

Proof. Since $M_5(ab0ba) = a(a-b)0(a-b)a$, fixed points satisfy the condition $a-b=b$ whence they are of the form $(2b)b0b(2b)$. Moreover, $M_5^2(ab0ba) = ab0ba$, so all the remaining elements of the form $ab0ba$ constitute 2-element M_5-orbits as required. □

Fact 2.4. *M_7-orbits are the following:*
1-element orbits (fixed points): $\{0\}$, $\{3210123\}$, $\{6420246\}$, $\{9630369\}$.
3-element orbits: 72 sets of the form

$$\big\{abc0cba,\ a(a-c)(b-c)0(b-c)(a-c)a,\ a(a-b+c)(a-b)0(a-b)(a-b+c)a\big\},$$

where $0 \leqslant c \leqslant b \leqslant a \leqslant 9$ *and* $abc0cba$ *is not a fixed point.*

Proof. We have $M_7(abc0cba) = a(a-c)(b-c)0(b-c)(a-c)a$ and $M_7^2(abc0cba) = a(a-b+c)(a-b)0(a-b)(a-b+c)a$ and $M_7^3(abc0cba) = abc0cba$. So fixed points are determined by solutions to the Diophantine system

$$\begin{cases} a-c=b \\ b-c=c, \end{cases} \quad\Longrightarrow\quad b=2c,\ a=3c$$

Now, to check whether we have 2-element orbits we solve the system

$$\begin{cases} a-b+c=b \\ a-b=c, \end{cases} \quad\Longrightarrow\quad b=2c,\ a=3c,$$

Since the only solutions are fixed points, we get a required contradiction. □

Remark 2.5. In particular, 3-element M_7-orbits of a regular form are

$$\big(kk(k-1)0(k-1)kk,\ k11011k,\ k(k-1)000(k-1)k\big),\ 1 \leqslant k \leqslant 9,$$
$$\big(k(k-1)(k-2)0(k-2)(k-1)k,\ k21012k,\ k(k-1)101(k-1)k\big),\ 2 \leqslant k \leqslant 9,\ k \neq 3,$$
$$\big(k(k-3)(k-4)0(k-4)(k-3)k,\ k41014k,\ k(k-1)303(k-1)k\big),\ 4 \leqslant k \leqslant 9,$$
$$\big(k(k-2)(k-4)0(k-4)(k-2)k,\ k42024k,\ k(k-2)202(k-2)k\big),\ 4 \leqslant k \leqslant 9,\ k \neq 6.$$

Fact 2.6. *M_9-orbits are the following:*
1-element orbits (fixed points): 22 sets of the form

$$\{(c+2d)(c+d)cd0dc(c+d)(c+2d)\},$$

where $0 \leqslant d \leqslant c \leqslant c+2d \leqslant 9$*,*
3-element orbits: 231 sets of the form

$$\big\{abcd0dcba,\quad a(a-d)(b-d)(b-c)0(b-c)(b-d)(a-d)a,$$
$$a(a-b+c)(a-b+c-d)(a-b)0(a-b)(a-b+c-d)(a-b+c)a\big\},$$

where $0 \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$ *and* $abcd0dcba$ *is not a fixed point.*

Proof. We have $M_9(abcd0dcba) = a(a-d)(b-d)(b-c)0(b-c)(b-d)(a-d)a$, $M_9^2(abcd0dcba) = a(a-b+c)(a-b+c-d)(a-b)0(a-b)(a-b+c-d)(a-b+c)a$ and $M_9^3(abcd0dcba) = abcd0dcba$. Now, fixed points are solutions to the Diophantine system

$$\begin{cases} a-d=b \\ b-d=c \\ b-c=d \end{cases} \iff \begin{cases} a=c+2d \\ b=c+d \end{cases}$$

whereas 2-element orbits can be determined by solving the system

$$\begin{cases} a-b+c=b \\ a-b+c-d=c \\ a-b=d \end{cases} \iff \begin{cases} a-b=d \\ d+c=b \end{cases} \iff \begin{cases} a=c+2d \\ b=c+d, \end{cases}$$

But the only elements satisfying this system are fixed points, a contradiction. □

Notation. Since elements we consider became too long in writing, from now on we shall use a briefer notation. Namely, we shall write all digits explicitly up to the middle 0 which will be bold. The remaining digits, except for the outermost, will be dotted.

Fact 2.7. *M_{11}-orbits are the following:*
1-element orbits (fixed points): $\{0\}$, $\{54321\mathbf{0}\ldots5\}$,
5-element orbits: 400 sets of the form

$$\{abcde\mathbf{0}\ldots a,\ a(a-e)(b-e)(b-d)(c-d)\mathbf{0}\ldots a,$$
$$a(a-c+d)(a-c+d-e)(a-b+d-e)(d-e)\mathbf{0}\ldots a,$$
$$a(a-d+e)(a-c+e)(b-c+e)(b-c)\mathbf{0}\ldots a,$$
$$a(a-b+c)(a-b+c-d+e)(a-b+c-d)(a-b)\mathbf{0}\ldots a\},$$

where $0 \leqslant e \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$ *and* $abcde\mathbf{0}\ldots a$ *is not a fixed point.*

Proof. The system of Diophantine equations providing fixed points is

$$\begin{cases} a-e=b \\ b-e=c \\ b-d=d \\ c-d=e, \end{cases} \implies d=2e, c=3e, b=4e, a=5e \implies e\in\{0,1\}$$

Similarly as in previous proofs, for $k=2,3,4$ the only solutions to each equation $M_{11}^k(abcde\mathbf{0}\dots a) = abcde\mathbf{0}\dots a$ are fixed points, which gives the statement. □

The trend that we could observe in all cases considered so far changes radically for $n=13$.

Fact 2.8. *M_{13}-orbits are the following:*
1-element orbits (fixed points): $\{0\}$, $\{654321\mathbf{0}\dots 6\}$,
2-element orbits:

$$\{976421\mathbf{0}\dots 9, 986542\mathbf{0}\dots 9\}, \quad \{322100\mathbf{0}\dots 3, 332221\mathbf{0}\dots 3\},$$
$$\{644200\mathbf{00}\dots 6, 664442\mathbf{0}\dots 6\}, \quad \{966300\mathbf{0}\dots 9, 996663\mathbf{0}\dots 9\},$$

3-element orbits: 11 sets of the form

$$\begin{aligned}\{&(2c-2f)(2c-e-f)c(e+f)ef\mathbf{0}\dots(2c-2f),\\ &(2c-2f)(2c-3f)(2c-e-2f)(2c-2e-f)(c-e)(c-e-f)\mathbf{0}\dots(2c-2f),\\ &(2c-2f)(c+e-f)(c+e-2f)(c+e-3f)(c-2f)(e-f)\mathbf{0}\dots(2c-2f)\},\end{aligned}$$

where $0\leqslant f\leqslant e\leqslant c\leqslant 2c-e-f\leqslant 2(c-f)\leqslant 9$ *and* $c-e-f\geqslant 0$
6-element orbits: 827 sets of the form

$$\begin{aligned}&\{abcdef\mathbf{0}\dots a,\ a(a-f)(b-f)(b-e)(c-e)(c-d)\mathbf{0}\dots a,\\ &a(a-c+d)(a-c+d-f)(a-c+e-f)(b-c+e-f)(e-f)\mathbf{0}\dots a,\\ &a(a-e+f)(a-c+d-e+f)(a-b+d-f+f)(a-b+d-e)(d-e)\mathbf{0}\dots a,\\ &a(a-d+e)(a-d+f)(b-d+f)(b-c+f)(b-c)\mathbf{0}\dots a\},\\ &a(a-c+b)(a-b+c-d+e)(a-b+c-d+e-f)(a-b+c-d)(a-b)\mathbf{0}\dots a\},\end{aligned}$$

where $0\leqslant f\leqslant e\leqslant d\leqslant c\leqslant b\leqslant a\leqslant 9$ *and* $abcdef\mathbf{0}\dots a$ *does not satisfy any of the above conditions.*

Proof. The statement follows from straightforward calculations. Let us only note that the only elements satisfying $M_{13}^4(\alpha)=\alpha$ belong either to 1- or 2-element M_{13}-orbits and the only elements satisfying $M_{13}^5(\alpha)=\alpha$ belong to 1-element orbits (i.e. are fixed points). □

Remark 2.9. We have verified numerically how many elements generate each type of the orbit (that is for how many β the number $M_{13}^k(\beta)$ belongs to some orbit for some k). So, fixed points are generated by 8370236170 numbers, 2-element orbits are generated by 60489967824 numbers, 3-element orbits are generated by 104611083720 numbers and finally 6-element orbits are generated by the remaining 9826528712286 numbers.

Fact 2.10. *M_{15}-orbits are the following:*
1-element orbits (fixed points): 44 sets of the form

$$\{(e+f+2g)(e+f+g)(e+f)(e+g)efg\mathbf{0}\ldots(e+f+2g)\}$$

where $0 \leqslant g \leqslant f \leqslant e \leqslant e+f+2g \leqslant 9$.
2-element orbits: 342 sets of the form

$$\{ab(b-g)(a-b+e)efg\mathbf{0}\ldots a,\ a(a-g)(b-g)(b-f)(b-f-g)(b-e-g)(a-b)\mathbf{0}\ldots a\}$$

where $0 \leqslant g \leqslant f \leqslant e \leqslant a-b+e \leqslant b-g \leqslant b \leqslant a \leqslant 9$ *and* $ab(b-g)(a-b+e)efg\mathbf{0}\ldots a$ *is not a fixed point.*
4-element orbits: 2678 sets of the form

$$\begin{aligned}\{&abcdefg\mathbf{0}\ldots a,\ a(a-g)(b-g)(b-f)(c-f)(c-e)(d-e)\mathbf{0}\ldots a,\\ &a(a-d+e)(a-d+e-g)(a-c+e-g)(b-c+e-g)(b-c+f-g)(b-c)\mathbf{0}\ldots a,\\ &a(a-b+c)(a-b+c-d+e)(a-b+c-d+e-f+g)(a-b+c-d+e-f)(a-b+c-d)(a-b)\mathbf{0}\ldots a\},\end{aligned}$$

where $0 \leqslant g \leqslant f \leqslant e \leqslant d \leqslant c \leqslant b \leqslant c \leqslant 9$ *and* $abcdefg\mathbf{0}\ldots a$ *does not satisfy any of the above conditions.*

Proof. As before, the statement follows from straightforward calculations. Let us only note that the only elements satisfying the equation $M_{15}^3(\alpha) = \alpha$ are fixed points. $\square$

Fact 2.11. *M_{17}-orbits are the following:*
1-element orbits (fixed points):

$$\begin{gathered}\{0\} \qquad \{444332100\mathbf{0}\ldots 4\},\ \{888664200\mathbf{0}\ldots 8\},\\ \{432211110\mathbf{0}\ldots 4\},\ \{876543210\mathbf{0}\ldots 8\},\ \{864422220\mathbf{0}\ldots 8\},\end{gathered}$$

2-element orbits: 32 sets of the form

$$\begin{aligned}\Big\{&(2d-2g+2h)(d+e-2g+2h)(d+e-2g+h)e(2g-h)gh\mathbf{0}\ldots(2d-2g+2h),\\ &(2d-2g+2h)(2d-2g+h)(d+e-2g+h)(d+e-3g+2h)\\ &\quad(d+e-3g+h)(d+e-4g+2h)(d-2g+h)(d-e)\mathbf{0}\ldots(2d-2g+2h)\}\end{aligned}$$

where $0 \leqslant h \leqslant g \leqslant 2g-h \leqslant e \leqslant d \leqslant 2d-2g+2h \leqslant 9$,
4-element orbits: 6060 sets of the form

$$\begin{aligned}\Big\{&abcdefgh\mathbf{0}\ldots a,\ a(a-h)(b-h)(b-g)(c-g)(c-f)(d-f)(d-e)\mathbf{0}\ldots a,\\ &a(a-d+e)(a-d+e-h)(a-d+f-h)(b-d+f-h)(b-c+f-h)(b-c+f-g)(b-c)\mathbf{0}\ldots a,\\ &a(a-b+c)(a-b+c-d+e)(a-b+c-d+e-f+g)\\ &\quad(a-b+c-d+e-f+g-h)(a-b+c-d+e-f)(a-b+c-d)(a-b)\mathbf{0}\ldots a\Big\}.\end{aligned}$$

where $0 \leqslant h \leqslant g \leqslant f \leqslant e \leqslant d \leqslant c \leqslant b \leqslant c \leqslant 9$ *and* $abcdefgh\mathbf{0}\ldots a$ *does not satisfy any of the above conditions.*

Proof. The statement follows from straightforward calculations. Let us only note that the only elements satisfying the equation $M_{17}^3(\alpha) = \alpha$ are fixed points. $\square$

Now let us notice some general properties of fixed points.

Fact 2.12.

1. *Let* $\alpha = a_1 \dots a_n 0 a_n \dots a_1$, *where* $0 \leqslant a_n \leqslant \dots \leqslant a_1 \leqslant 9$. *Then* α *is a fixed point of* M_{2n+1} *if and only if*

$$\begin{cases} a_1 - a_n = a_2 \\ a_2 - a_n = a_3 \\ \vdots \\ a_k - a_{n-k+1} = a_{2k} \\ a_{k+1} - a_{n-k+1} = a_{2k+1} \\ \vdots \\ a_{\lceil \frac{n}{2} \rceil} - a_{1+\lceil \frac{n}{2} \rceil} = a_n \end{cases} \tag{3}$$

   where $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$.
2. *Elements of the form*

$$(nc)((n-1)c)\dots c0c(2c)\dots(nc), \tag{4}$$

   where $c \in \{0, 1, \dots, 9\}$ *and* $nc \leqslant 9$ *are fixed points of* M_{2n+1}. *In particular, for* $n > 9$ *the only fixed point of* M_{2n+1} *of the form (4) is 0. If* $5 \leqslant n \leqslant 9$ *then* M_{2n+1} *has two fixed points of the form (4), for* $n = 4$ *there are three such points, for* $n = 3$ *– four such points, for* $n = 2$ *– five of them and for* $n = 1$ *– ten fixed points of the form (4).*
3. *If* $n = 1 \pmod 3$ *then* M_n *possesses fixed points which are not of the form (4).*

Proof. 1. and 2. are straightforward. So as to 3., observe that if $n = 3N + 1$, then two equations in the system (3), namely

$$a_{N+1} - a_{n-N+1} = a_{2N+1} \qquad \text{and} \qquad a_{N+1} - a_{n-N} = a_{2N+2},$$

are equivalent, which provides one more parameter in the solution of (3), whence more fixed points of M_n. □

Examining the form of orbits we suspect that the following are true:

Conjecture 1. Each number of the form $a_1 \dots a_n 0 a_n \dots a_1$, $0 \leqslant a_n \leqslant \dots \leqslant a_1 \leqslant 9$ belongs to some M_{2n+1}-orbit.

Conjecture 2. For odd n, each M_n-orbit has at most $\frac{n-1}{2}$ elements.

2.2. M_n-orbits for even n

In comparison to odd n, the number and form of M_n-orbits for even n is surprisingly measly. But thanks to this it was possible to describe M_n-orbits for every even n.

First let us collect the results obtained by straightforward calculations (numerical) for every even $n \leqslant 20$.

Notation. Since digits appear in bunches, we shall use the subscript to denote how many consecutive occurrences of a given digit we have. For instance, a number $(2b)(2b)bb00bb(2b)(2b)$ will be denoted by $(2b)_2b_20_2b_2(2b)_2$.

Fact 2.13. *For even $n \leqslant 20$ we have the following M_n-orbits:*

1. $n = 2$: *1-element orbit:* $\{0\}$.
2. $n = 4$: *1-element orbit:* $\{0\}$.
3. $n = 6$: *1-element orbits:* $\{a_20_2a_2\}$, *where* $0 \leqslant a \leqslant 9$,
4. $n = 8$: *1-element orbit:* $\{0\}$.
5. $n = 10$: *1-element orbits:* $\{(2b)_2b_20_2b_2(2b)_2\}$, *where* $0 \leqslant b \leqslant 4$,
2-element orbits:
$$\{a_2b_20_2b_2a_2,\quad a_2(a-b)_20_2(a-b)_2a_2\},$$
where $0 \leqslant b \leqslant a \leqslant 9$ *and* $a_2b_20_2b_2a_2$ *is not a fixed point.*
6. $n = 12$: *1-element orbits* $\{a_40_4a_4\}$, *where* $0 \leqslant a \leqslant 9$.
7. $n = 14$: *1-element orbits:* $\{(3b)_2(2b)_2b_20_2b_2(2b)_2(3b)_2\}$, *where* $0 \leqslant b \leqslant 3$,
3-element orbits:
$$\{a_2b_2c_20_2c_2b_2a_2,\ a_2(a-c)_2(b-c)_20_2(b-c)_2(a-c)_2a_2,$$
$$a_2(a-b+c)_2(a-b)_20_2(a-b)_2(a-b+c)_2a_2\},$$
where $0 \leqslant c \leqslant b \leqslant a \leqslant 9$ *and* $a_2b_2c_20_2c_2b_2a_2$ *is not a fixed point.*
8. $n = 16$: *1-element orbit:* $\{0\}$.
9. $n = 18$: *1-element orbits:* $\{(2d+c)_2(d+c)_2c_2d_20_2d_2c_2(d+c)_2(2d+c)_2\}$, *where* $0 \leqslant d \leqslant c \leqslant 2d+c \leqslant 9$,
3-element orbits
$$\{a_2b_2c_2d_20_2d_2c_2b_2a_2,$$
$$a_2(a-d)_2(b-d)_2(b-c)_20_2(b-c)_2(b-d)_2(a-d)_2a_2,$$
$$a_2(a-b+c)_2(a-b+c-d)_2(a-b)_20_2(a-b)_2(a-b+c-d)_2(a-b+c)_2a_2\},$$
where $0 \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$ *and* $a_2b_2c_2d_20_2d_2c_2b_2a_2$ *is not a fixed point.*
10. $n = 20$: *1-element orbits:* $\{(2b)_4b_40_4b_4(2b)_4\}$, *where* $1 \leqslant b \leqslant 4$,
2-element orbits:
$$\{a_4b_40_4b_4a_4,\ a_4(a-b)_40_4(a-b)_4a_4\},$$
where $0 \leqslant b \leqslant a \leqslant 9$ *and* $a_4b_40_4b_4a_4$ *is not a fixed point.* □

The theorem below provides the full description of M_n-orbits for even n, as every even natural number can be uniquely written in the form 2^kr, where $k, r \in \mathbb{N}$ and r is odd.

Theorem 2.14.

1. *For every natural k and each $\alpha < 10^{2^k}$ we have $M_{2^k}^{k+1}(\alpha) = 0$, which means that the only M_{2^k}-orbit is $\{0\}$.*
2. *For every $r \in 2\mathbb{N}+1$ and every natural k each $M_{2^k r}$-orbit is determined by a unique M_r-orbit. More precisely, a set $\{\beta_1^{(k)}, .., \beta_s^{(k)}\}$ is an $M_{2^k(2n+1)}$-orbit if and only if the set $\{\beta_1, .., \beta_s\}$ is an M_{2n+1}-orbit, where for $\alpha := a_1 \dots a_n$ we define $\alpha^{(k)} := (a_1)_{2^k} \dots (a_n)_{2^k}$.*

Proof. We first show the following: Let $k, r, A \in \mathbb{N}$ be such that r is odd and $A < 10^{2^k r}$ with digits $0 \leqslant a_{2^k r} \leqslant \dots \leqslant a_1 \leqslant 9$. Then for every natural $m \leqslant k$ there exist numbers $0 \leqslant a_{2^{k-m}r}^{(m)} \leqslant \dots \leqslant a_1^{(m)} \leqslant 9$ such that

$$M^m(A) = \left(a_1^{(m)}\right)_{2^{m-1}} \cdots \left(a_{2^{k-m}r}^{(m)}\right)_{2^{m-1}} \left(a_{2^{k-m}r}^{(m)}\right)_{2^{m-1}} \cdots \left(a_1^{(m)}\right)_{2^{m-1}}$$

We take $a_i^{(1)} := a_i - a_{2^k r - i + 1}$ whence the statement is true for $m = 1$. For every $m < k$ we take $a_i^{(m+1)} := a_i^{(m)} - a_{2^{k-m}r-i+1}^{(m)}$, so the result follows by induction.

Now, if $r = 1$ then $M_{2^k}^k(A) = a_{2^k}$ for some a, whence $M_{2^k}^{k+1}(A) = 0$ which gives 1.

For $r = 2n+1 > 1$ for $m = k$ we get

$$M_{2^k r}^k(A) = \left(a_1^{(k)}\right)_{2^{k-1}} \cdots \left(a_{2n+1}^{(k)}\right)_{2^{k-1}} \left(a_{2n+1}^{(k)}\right)_{2^{k-1}} \cdots \left(a_1^{(k)}\right)_{2^{k-1}}$$

whence

$$M_{2^k r}^{k+1}(A) = \left(a_1^{(k+1)}\right)_{2^k} \cdots \left(a_n^{(k+1)}\right)_{2^k} 0_{2^k} \left(a_n^{(k+1)}\right)_{2^k} \cdots \left(a_1^{(k+1)}\right)_{2^k}$$

Now, it follows by induction that for every $m > k$ we have

$$M_{2^k r}^m(A) = \left(a_1^{(m)}\right)_{2^k} \cdots \left(a_n^{(m)}\right)_{2^k} 0_{2^k} \left(a_n^{(m)}\right)_{2^k} \cdots \left(a_1^{(m)}\right)_{2^k}$$

where for $1 \leqslant s \leqslant \lfloor \frac{n}{2} \rfloor$ we have

$$a_{2s}^{(m+1)} = a_s^{(m)} - a_{n-s+1}^{(m)}, \qquad a_{2s+1}^{(m+)} = a_{s+1}^{(m)} - a_{n-s+1}^{(m)}$$

and $a_{n+1}^{(m+1)} = 0$. That means that the form of $M^m(A)$ does not depend on k but only on r. Moreover, $M_{2^k r}^m$ acts on the bunches of 2^k consecutive digits in exactly the same way as M_r^m act on single digits, which finishes the proof. □

Remark 2.15. Let us show on examples how the above theorem works.

Since by Fact 2.2 the operator M_3 has only 1-element orbits of the form $\{a0a\}$ (there are ten of them), for each k there are only ten $M_{3 \cdot 2^k}$-orbits of the form $\{a_{2^k} 0_{2^k} a_{2^k}\}$ (see Fact 2.13 for $n = 6, 12$).

For $r = 5$ by Fact 2.3 and the above theorem we obtain that $M_{5 \cdot 2^k}$-orbits are the following:

1-element orbits: $\{(2b)_{2^k} b_{2^k} 0_{2^k} b_{2^k} (2b)_{2^k}\},\ 0 \leqslant b \leqslant 4$,

2-element orbits: $\{a_{2^k} b_{2^k} 0_{2^k} b_{2^k} a_{2^k},\ a_{2^k} (a-b)_{2^k} 0_{2^k} (a-b)_{2^k} a_{2^k}\}$

where $0 \leqslant b \leqslant a \leqslant 9$ and $a \neq 2b$ (see Fact 2.13 for $n = 10, 20$).

3. Permutational Kaprekar's transformations

In this section we are going to consider transformations which generalizes the idea of Kaprekar's transformation in some other direction than symmetric ones, namely

$$T_{(p,n)}(\alpha) := \sum_{k=1}^{n}(a_k - a_{p(k)})10^{k-1} = a_n \dots a_2 a_1 - a_{p(n)} \dots a_{p(2)} a_{p(1)}$$

where α is a number with digits $0 \leqslant a_1 \leqslant a_2 \leqslant \dots \leqslant a_n \leqslant 9$ and p is a permutation in S_n (that is defined on the set $\{1, \dots, n\}$). Note that the classical Kaprekar's transformation T_n is of that form for $p = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$. Moreover, for each permutation p and every number α we have two inequalities:

$$0 \leqslant T_{(p,n)}(\alpha) \leqslant T_n(\alpha).$$

Indeed, $a_n \dots a_1$ is the greatest number of the form $a_{p(n)} \dots a_{p(1)}$ and $a_1 \dots a_n$ is the smallest one.

The rest of this section is devoted to examination of orbits of transformations

$$Q_n(\alpha) := \sum_{k=1}^{n-2}(a_k - a_{n-k+1})10^{k-1} + (a_{n-1} - a_1)10^{n-2} + (a_n - a_2)10^{n-1}, \quad n \geqslant 3.$$

Note that $Q_n(\alpha) = T_{(p,n)}(\alpha)$ where $p = \begin{pmatrix} 1 & 2 & \dots & n-2 & n-1 & n \\ n & n-1 & \dots & 3 & 1 & 2 \end{pmatrix}$. Moreover, if for some α we have $a_1 = a_2$, then $Q_n(\alpha) = T_n(\alpha)$. So actually to obtain Q_n we modified T_n the least we could.

Most of the results we present were obtained numerically by straightforward calculations. Note however, that we can prove some general facts as well.

Notation. Since now it is not enough to deal only with digits, we shall also use the symbol $\times$ to denote multiplication. So, as before, AB denotes the number with digits A and B, while $9 \times AB$ denotes the product of 9 and the number AB. On the other hand, to simplify a little, when there is not risk of confusion we shall omit $\times$, like in $(a-b)10$.

Fact 3.1. *Q_3-orbits contain only numbers of the form $9 \times BA$ where $0 \leqslant B \leqslant A \leqslant 9$.*

Proof. Let $\alpha \in \mathbb{N}$ be a number with digits $0 \leqslant c \leqslant b \leqslant a \leqslant 9$. Then

$$\begin{aligned} Q_3(\alpha) &= a(10^2 - 1) + b(10 - 10^2) + c(1 - 10) = \\ &= 9((a-b)10 + a - c) = 9 \times BA, \end{aligned}$$

where $A := a - c,\ B := a - b$, whence $0 \leqslant B \leqslant A \leqslant 9$. □

Fact 3.2. *Q_3 posses only one fixed point, namely 0, and one 3-element orbit $\{135, 216, 405\}$. The sum of digits of each number in this orbit equals 9.* □

Fact 3.3. *Q_4-orbits contain only numbers of the form*

$$9 \times B(A+B-C)A, \tag{5}$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant A+B-C \leqslant 9$ *or*

$$9 \times (B+1)(A+B-C-10)A, \tag{6}$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant 9 < A+B-C$.

Proof. Let $\alpha \in \mathbb{N}$ be a number with digits $0 \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$. Then

$$\begin{aligned} Q_4(\alpha) &= a(10^3-1)+b(10^2-10)-c(10^3-10)-d(10^2-1) = \\ &= 9\big(a(10^2+10+1)+b10-c(10^2+10)-d(10+1)\big) = \\ &= 9\big((a-c)10^2+(a-c+b-d)10+a-d\big) = \\ &= 9\big((a-c+1)10^2+(a-c+b-d-10)10+a-d\big). \end{aligned}$$

If we set $A := a-d$, $B := a-c$, $C := a-b$ then $0 \leqslant A \leqslant C \leqslant B \leqslant 9$ and $a-c+b-d = A+B-C$, whence we obtain the statement. □

Fact 3.4.

1. *There are only three Q_4-orbits:* $\{0\}$, $\{2187, 6543\}$ *and* $\{3285, 5274\}$. *The sum of digits of each number in nontrivial orbits equals 18.*
2. *By formulas (5) and (6) we have the following:*

$$Q_4(a(a-1)(a-2)(a-3)) = 2187, \quad \textit{for every} \quad 3 \leqslant a \leqslant 9,$$

$$Q_4(aa(a-b)(a-b)) = b(b-1)(9-b)(10-b) \quad \textit{for every} \quad 1 \leqslant b \leqslant a \leqslant 9,$$

$$Q_4^4(aa(a-5)(a-5)) = 6543,$$

$$\begin{aligned} &Q_4^2(aa(a-b)(a-b)) = \\ &\qquad = \begin{cases} (2b-10)(2b-11)(20-2b)(19-2b), & \textit{if } b = 6,7,8,9, \\ (10-2b)(9-2b)(2b)(2b-1), & \textit{if } b = 1,2,3,4, \end{cases} \end{aligned}$$

$$= \begin{cases} 2187, \textit{ if } b = 4,6, \\ 8721, \textit{ if } b = 1,9, \\ 6543, \textit{ if } b = 2,8, \\ 4365, \textit{ if } b = 3,7, \end{cases}$$

3. *The number of iterations of Q_4 which leads to elements in orbits can be pretty large, for instance the smallest n for which $Q_4^n(7092)$ belongs to some Q_4-orbit is 11 (and we have $Q_4^{11}(7092) = 2187$).* □

Fact 3.5. *Nontrivial Q_5-orbits contain only elements of the form*

$$(9-d)(b-e-1)9(9-b+d)(e+1), \tag{7}$$

where $0 \leqslant e \leqslant d \leqslant b \leqslant 9$ and $b-e \geqslant 1$. The sum of digits of each number in each orbit equals 27. In particular, the only fixed point of Q_5 is 0.

Proof. Let $\alpha \in \mathbb{N}$ be a number with digits $0 \leqslant e \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$. Then

$$\begin{aligned} Q_5(\alpha) &= (a-d)10^4 + (b-e)10^3 - (b-d)10 - (a-e) = \\ &= (a-d)10^4 + (b-e-1)10^3 + 9\cdot 10^2 + (9-b+d)10 + 10 - a + e. \end{aligned}$$

Now, if $b-e=0$, then $b=c=d=e$ and $0 \leqslant a-b \leqslant 9$ whence we obtain $Q_5(\alpha) = (a-b)(10^4-1)$. By straightforward calculations we obtain that all ten numbers of this form generate Q_5-orbit $\{52974, 54963\}$. Both elements are of the form (7) as required. Now, if $b-e \geqslant 1$, then $a-e \geqslant 1$ whence formula for $Q_5(\alpha)$ give its digits. Note that 9 is a digit in $Q_5(\alpha)$ for every α, so in particular for elements in Q_5-orbits there must be $a=9$ (as this is the greatest digit) whence we get (7) as required.

Now, let α be a nontrivial fixed point, that is $Q(\alpha)=\alpha$. Then $a=9$ and $b-e \geqslant 1$ whence $e+1 \neq e$ and $e+1 \neq b$. Moreover $9-d \neq d$. Thus there are two possibilities:

1) $e+1=d$. Since $9-b+d > d$ then $b-e-1=e \iff b=2e+1$. Now, the remaining two digits are b, c. If we assume that $9-d=b \iff 9-b+d=2d$ then $c=2d=2e+2>2e+1=b$, a contradiction. If we assume that $9-b+d=b \iff 9-d=2b$, then $2b=c$, a contradiction. So this case is impossible. 2) $e+1=c$.

Since $b-e-1<b$ then it has to be equal either to e or d. In the first case we obtain $b=2e+1$, $9-b+d=d \iff b=9 \iff e=4$ and $9-d=b=9 \iff d=0<e$, a contradiction. If $b-e-1=d$ then $9-b+d=9-c \leqslant 9-d$ whence $9-d=b$, $9-b+d=e \iff e=2d \iff d=e=0 \iff b=1$. But then $9-d \neq b$ which makes this case impossible and finishes the proof. □

Examining the results of numerical calculations of Q_n-orbits for all $n \leqslant 15$ we have noticed some regularities which gave birth to some general facts. The first theorem below concerns fixed points, the second – some 3-element orbits.

Theorem 3.6.

1.

$$\underbrace{5\ldots5}_{k\text{-times}}\,4\,\underbrace{9\ldots9}_{(k+1)\text{-times}}\,\underbrace{4\ldots4}_{k\text{-times}}\,5$$

is a fixed point of Q_{3k+3} for every $k \in \mathbb{N}$.

2.

$$66\,\underbrace{3\ldots3}_{k\text{-times}}\,08\,\underbrace{6\ldots6}_{k\text{-times}}\,52$$

is a fixed point of Q_{2k+6} for every $k \in \mathbb{N}_0$.

3.

$$\underbrace{9\ldots9}_{(k+1)\text{-}times}\ 75308642\,\underbrace{0\ldots0}_{k\text{-}times}\,1 \quad \textit{and} \quad 675\,\underbrace{3\ldots3}_{k\text{-}times}\,2087\,\underbrace{6\ldots6}_{k\text{-}times}\,442$$

are fixed points of Q_{2k+10} for every $k \in \mathbb{N}$ and $k \in \mathbb{N}_0$, respectively.

4.

$$\underbrace{9\ldots9}_{(k+1)\text{-}times}\ 750842\,\underbrace{0\ldots0}_{k\text{-}times}\,1$$

is a fixed point of Q_{2k+8} for every $k \in \mathbb{N}$.

Proof. 1. For every $k \in \mathbb{N}$ we have

$$Q_{3k+3}\left(\underbrace{5\ldots5}_{k\text{-}times}4\ \underbrace{9\ldots9}_{(k+1)\text{-}times}\underbrace{4\ldots4}_{k\text{-}times}5\right) = \underbrace{9\ldots9}_{(k+1)\text{-}times}\underbrace{5\ldots5}_{(k+1)\text{-}times}\underbrace{4\ldots4}_{(k+1)\text{-}times} - \underbrace{4\ldots4}_{(k+1)\text{-}times}\underbrace{5\ldots5}_{(k+1)\text{-}times}\underbrace{9\ldots9}_{(k+1)\text{-}times} = \underbrace{5\ldots5}_{k\text{-}times}4\ \underbrace{9\ldots9}_{(k+1)\text{-}times}\ \underbrace{4\ldots4}_{k\text{-}times}5$$

2. For every $k \in \mathbb{N}_0$ we have

$$Q_{2k+6}\left(66\,\underbrace{3\ldots3}_{k\text{-}times}\,08\,\underbrace{6\ldots6}_{k\text{-}times}\,52\right) = 86\,\underbrace{6\ldots6}_{k\text{-}times}\,65\,\underbrace{3\ldots3}_{k\text{-}times}\,20 - 20\,\underbrace{3\ldots3}_{k\text{-}times}\,56\,\underbrace{6\ldots6}_{k\text{-}times}\,68 = 66\,\underbrace{3\ldots3}_{k\text{-}times}\,08\,\underbrace{6\ldots6}_{k\text{-}times}\,52$$

3. For every $k \in \mathbb{N}$ we have

$$Q_{2k+10}\left(\underbrace{9\ldots9}_{(k+1)\text{-}times}\ 75308642\,\underbrace{0\ldots0}_{k\text{-}times}\,1\right) = \underbrace{9\ldots9}_{(k+1)\text{-}times}\ 87654321\ \underbrace{0\ldots0}_{(k+1)\text{-}times} - \underbrace{0\ldots0}_{(k+1)\text{-}times}\ 12345678\ \underbrace{9\ldots9}_{(k+1)\text{-}times} = \underbrace{9\ldots9}_{(k+1)\text{-}times}\ 75308642\ \underbrace{0\ldots0}_{k\text{-}times}\,1$$

For every $k \in \mathbb{N}_0$ we have

$$Q_{2k+10}\left(675\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,442\right) = \begin{array}{r} 877\,\underbrace{6\ldots6}_{k\text{-times}}\,6544\,\underbrace{3\ldots3}_{k\text{-times}}\,220 \\ -\,202\,\underbrace{3\ldots3}_{k\text{-times}}\,4456\,\underbrace{6\ldots6}_{k\text{-times}}\,778 \\ \hline 675\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,442 \end{array}$$

4. For every $k \in \mathbb{N}$ we have

$$Q_{2k+8}\left(\underbrace{9\ldots9}_{(k+1)\text{-times}}\,750842\,\underbrace{0\ldots0}_{k\text{-times}}\,1\right) = \begin{array}{r} \underbrace{9\ldots9}_{(k+1)\text{-times}}\,875421\,\underbrace{0\ldots0}_{(k+1)\text{-times}} \\ -\,\underbrace{0\ldots0}_{(k+1)\text{-times}}\,124578\,\underbrace{9\ldots9}_{(k+1)\text{-times}} \\ \hline \underbrace{9\ldots9}_{(k+1)\text{-times}}\,750842\,\underbrace{0\ldots0}_{k\text{-times}}\,1 \end{array}$$

which finishes the proof.

$\square$

Theorem 3.7.

1. Q_{2k+8}, $k \in \mathbb{N}_0$ possesses the following 3-element orbit:

$$\left\{56\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,44,\quad 67\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,42,\quad 675\,\underbrace{3\ldots3}_{k\text{-times}}\,17\,\underbrace{6\ldots6}_{k\text{-times}}\,442\right\}$$

2. Q_{2k+10}, $k \in \mathbb{N}_0$ possesses the following 3-element orbit:

$$\left\{64\,\underbrace{3\ldots3}_{(k+1)\text{-times}}\,1088\,\underbrace{6\ldots6}_{(k+1)\text{-times}}\,54,\quad 78\,\underbrace{3\ldots3}_{(k+1)\text{-times}}\,2087\,\underbrace{6\ldots6}_{(k+1)\text{-times}}\,22,\quad 6855\underbrace{3\ldots3}_{k\text{-times}}2\,\underbrace{6\ldots6}_{(k+1)\text{-times}}\,4432\right\}$$

Proof. 1. For every $k \in \mathbb{N}_0$ we have

$$Q_{2k+8}\left(56\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,44\right) = \begin{array}{r} 87\,\underbrace{6\ldots6}_{k\text{-times}}\,6544\,\underbrace{3\ldots3}_{k\text{-times}}\,20 \\ -\,20\,\underbrace{3\ldots3}_{k\text{-times}}\,4456\,\underbrace{6\ldots6}_{k\text{-times}}\,78 \\ \hline 67\,\underbrace{3\ldots3}_{k\text{-times}}\,2087\,\underbrace{6\ldots6}_{k\text{-times}}\,42, \end{array}$$

$$Q_{2k+8}\left(67\,\underbrace{3\ldots3}_{k\text{-}times}\,2087\,\underbrace{6\ldots6}_{k\text{-}times}\,42\right) = \begin{array}{r} 877\,\underbrace{6\ldots6}_{k\text{-}times}\,64\,\underbrace{3\ldots3}_{k\text{-}times}\,220 \\ -\,202\,\underbrace{3\ldots3}_{k\text{-}times}\,46\,\underbrace{6\ldots6}_{k\text{-}times}\,778 \\ \hline 675\,\underbrace{3\ldots3}_{k\text{-}times}\,17\,\underbrace{6\ldots6}_{k\text{-}times}\,442, \end{array}$$

$$Q_{2k+8}\left(675\,\underbrace{3\ldots3}_{k\text{-}times}\,17\,\underbrace{6\ldots6}_{k\text{-}times}\,442\right) = \begin{array}{r} 77\,\underbrace{6\ldots6}_{k\text{-}times}\,6544\,\underbrace{3\ldots3}_{k\text{-}times}\,21 \\ -\,21\,\underbrace{3\ldots3}_{k\text{-}times}\,4456\,\underbrace{6\ldots6}_{k\text{-}times}\,77 \\ \hline 56\,\underbrace{3\ldots3}_{k\text{-}times}\,2087\,\underbrace{6\ldots6}_{k\text{-}times}\,44, \end{array}$$

2. For every $k \in \mathbb{N}_0$ we have

$$Q_{2k+10}\left(64\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,1088\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,54\right) = \begin{array}{r} 88\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,6544\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,10 \\ -\,10\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,4456\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,88 \\ \hline 78\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,2087\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,22, \end{array}$$

$$Q_{2k+10}\left(78\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,2087\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,22\right) = \begin{array}{r} 8877\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,2220 \\ -\,2022\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,7788 \\ \hline 6855\,\underbrace{3\ldots3}_{k\text{-}times}\,2\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,4432, \end{array}$$

$$Q_{2k+10}\left(6855\,\underbrace{3\ldots3}_{k\text{-}times}\,2\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,4432\right) = \begin{array}{r} 86\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,5544\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,22 \\ -\,22\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,4455\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,68 \\ \hline 64\,\underbrace{3\ldots3}_{(k+1)\text{-}times}\,1088\,\underbrace{6\ldots6}_{(k+1)\text{-}times}\,54, \end{array}$$

$\square$

In the table below we collect all the numerical results we obtained. Here s denotes the sum of digits of every element in the given orbit, d stands for the length of an orbit and m gives the number of numbers that generate the given orbit.

Q_n-orbits	s	d	m
$n=2$			
{0}	0	1	100
$n=3$			
{0}	0	1	28
{135,216,405}	9	3	972
$n=4$			
{0}	0	1	10
{3285,5274}	18	2	1908
{2187,6543}	18	2	8082
$n=5$			
{0}	0	1	10
{52974,54936}	27	2	99990
$n=6$			
{0}	0	1	10
{660852}	36	1	10080
{549945}	36	1	204102
{350874,570852,669942,569943, 560844,460872,671742,561744}	27,36	8	785808
$n=7$			
{0}	0	1	10
{5729643,6519753,6609852,7809831, 8849421,7739532,6539553,6299964}	36, 45	8	9999990
$n=8$			
{0}	0	1	10
{66308652}	36	1	215040
{56208744,67208742,67517442}	36	3	99784950

Q_n-orbits	s	d	m
$n=9$			
{0}	0	1	10
{554999445}	54	1	34440
{763197642,764197542}	45	2	42954837
{752197743,764296542}	45	2	783607101
{652098753,784098531,885296421,776197332,764395542}	45	5	173403612
$n=10$			
{0}	0	1	10
{6633086652}	45	1	2520000
{9975084201}	45	1	41045760
{6752087442}	45	1	50793120
{6431088654,7832087622,6855264432}	45	3	1581982950
{5632087644,6732087642,6753176442}	45	3	8323658160
$n=11$			
{0}	0	1	10
{76421977542,76531976442}	54	2	1152074022
{77420987532,78542965431}	54	2	20024739790
{77530986432,78441975531,87430986522, 78641975331,87441975522}	54	5	78823186178
$n=12$			
{0}	0	1	10
{555499994445}	72	1	697950
{663330866652}	54	1	23950080
{999750842001}	54	1	556839360
{997530864201}	54	1	6771885120
{997510884201,997750842201,997550844201}	54	3	10397350260
{643110888654,787320876222,685552644432}	54	3	14282581632
{654310886544,783210887622,786552644322}	54	3	49942260696
{643310886654,783320876622,685532664432}	54	3	139166335086
{563320876644,673320876642,675331766442}	54	3	775875839646
$n=13$			
{0}	0	1	10
{7742109887532,8865429654321, 7763209876332,7854429655431}	63	4	510013982062
{7865309864331,8854319765421,8764209875322, 7865419754331,8744209875522}	63	5	9489986017928

Q_n-orbits	s	d	m
$n = 14$			
{0}	0	1	10
{66333308666652}	63	1	197765568
{99997508420001}	63	1	6034588560
{67533208766442}	63	1	87185978880
{99975308642001}	63	1	126071225280
{99753308664201}	63	1	516356961120
{99975108842001,99977508422001,99975508442001}	63	3	181449067800
{64311108888654,78773208762222,68555526444432}	63	3	270059528648
{99751108884201,99777508422201,99755508444201}	63	3	420203255472
{99755108844201,99775108842201,99775508442201}	63	3	829988923764
{64331108886654,78733208766222,68555326644432}	63	3	1186659773208
{65431108886544,78732108876222,78655526444322}	63	3	1935985009868
{65543108865444,78321108887622,78765526443222}	63	3	2182686460318
{99753108864201,99775308642201,99755308644201}	63	3	2316236914992
{65433108866544,78332108876622,78655326644322}	63	3	6805012345352
{64333108866654,78333208766622,68553326664432}	63	3	10677174402826
{56333208766644,67333208766642,67533317666442}	63	3	72458697798334
$n = 15$			
{0}	0	1	10
{555549999944445}	90	1	15165150
{776321098876332,886544296554321}	72	2	9882019583415
{775432098765432,785442098755431,886431098865321, 887653197643221,876543197654322}	72	5	102564269404575
{764432098765542,785422098775431,886543098654321, 886532098764321,886643197653321, 876433197665322,775433197665432}	72	7	887553695846850

In the table above some interesting properties and anomalies can be observed:

1. Trivial fixed points for Q_n are always generated by 10 numbers except for $n = 2, 3$.
2. Transformations with only trivial fixed points are $Q_3,\ Q_4,\ Q_5,\ Q_7,\ Q_{11}$ and Q_{13}.
3. Q_3, Q_5 and Q_7 are the only transformations possessing only two orbits.
4. The sum of digits of elements in each orbit is constant except for one orbit for Q_6 (for which it equals 27 or 36) and one for Q_7 (equals 36 or 45).
5. There is also stronger general property – namely the sum of digits of elements in all nontrivial orbits is also constant except for $Q_{3n},\ n = 2, 3, 4, 5$ and Q_7.

4. General Kaprekar's transformations

In previous sections we have generalized the idea of the classical Kaprekar's transformation using absolute value or allowing the second index to vary according to some permutation. Now we shall consider transformations that generalizes the idea of both previously mentioned ones.

Namely, again we shall consider numbers with at most n digits (filled with 0 at the beginning if necessary), but now in the original order.

Definition 4.1. *Let α be a number with at most n digits (filled with 0 at the beginning if necessary), that is*

$$\alpha = d_1 d_2 \dots d_n,\ 0 \leqslant d_i \leqslant 9$$

*and let σ, π be permutations of $\{1, \dots n\}$. Then the n-**th** (σ,π)-**general Kaprekar's transformation** $d_n^{(\sigma,\pi)}$ is the function defined in the following way*

$$d_n^{(\sigma,\pi)}(\alpha) := \sum_{k=1}^{n} |d_{\sigma(k)} - d_{\pi(k)}| 10^{n-k-1},$$

In the sequel we shall consider orbits for a special type of transformations of this kind, namely for σ being the identity permutation and $\pi_n = \begin{pmatrix} 1 & 2 & \dots & n-2 & n-1 & n \\ 2 & 3 & \dots & n-1 & n & 1 \end{pmatrix}$, i.e. π_n being the cycle $(1, 2, \dots, n)$, that is

$$D_n := d_n^{(id,\pi_n)} = \sum_{k=1}^{n-1} |d_k - d_{k+1}| 10^{n-k} + |d_n - d_1| \tag{8}$$

The permutations we consider were not chosen by accident. Note that transformations D_n are actually special cases of so-called Ducci's transformation (see [1–4, 15] for more details on Ducci's transformations).

Theorem 4.2.

1. *The only fixed point of D_n, $n \in \mathbb{N}$ is 0.*
2. *Transformations D_{2^n}, $n \in \mathbb{N}$ have only one orbit – the trivial one.*

Proof. 1. Let $\alpha = d_1 d_2 \dots d_n$, $0 \leqslant d_i \leqslant 9$ and let $D_n(\alpha) = \alpha$. Then

$$\begin{cases} |d_k - d_{k+1}| = d_k, & k \in \{1, 2, \dots, n-1\} \\ |d_n - d_1| = d_n. \end{cases} \tag{9}$$

There are two possibilities:

a. There exists $m \in \{1, \dots, n-1\}$ such that $d_m \geqslant d_{m+1}$. Then from (9) we get $d_m - d_{m+1} = d_m$ whence $d_{m+1} = 0$ and hence recursively for every $k \in \{1, \dots, n\}$ we have $d_k = 0$.
b. For every $k \in \{1, \dots, n-1\}$ we have $d_k < d_{k+1}$. Then $d_1 < d_n$ and from the last equation in (9) we get $d_1 = 0$, which recursively implies that $d_k = 0$ for every $k \in \{1, 2, \dots, n\}$.

So the only fixed point of D_n is 0.

2. It follows directly from the following theorem proven in [13]:
Theorem: Let T be the transformation $T\colon \mathbb{Z}^{2^n} \to \mathbb{Z}^{2^n}$ defined in the following way

$$\mathbb{Z}^{2^n} \ni (a_1, a_2, \dots, a_{2^n}) \mapsto (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_k - a_{k+1}|, \dots |a_{2^n} - a_1|)$$

If $a := (a_1, a_2, \dots, a_{2^n}) \in \mathbb{Z}^{2^n}$ is fixed then there exists $k := k(a) \in \mathbb{N}$ such that $T^k(a)$ is the zero vector. □

In the table below we collect the results on nontrivial D_n-orbits for $n \leqslant 10$ obtained numerically, where k stands for any nonzero digit.

D_n-orbits	Length
$n = 3$	
$\{0\}$	1
$\{kk, k0k, kk0\}$	3
$n = 4$	
$\{0\}$	1
$n = 5$	
$\{0\}$	1
$\{kk, k0k, kkkk, k000k, k00k0, k0kkk, kk000, k00k,$ $kk0kk, kk00, k0k00, kkk0k, kk0, k0k0, kkkk0\}$	15
$n = 6$	
$\{0\}$	1
$\{kk0kk, k0kk0k, kk0kk0\}$	3
$\{k0k, kkkk, k000k, kk00kk, k0k00, kkkk00\}$	6
$\{k0k0, kkkk0, k000k0, k00kkk, k0k000, kkk00k\}$	6
$n = 7$	
$\{0\}$	1
$\{kk, k0k, kkkk, k000k, kk00kk, k0k0k0k, kkkkkk0\}$	7
$\{kk0, k0k0, kkkk0, k000k0, kk00kk0, k0k0kk, kkkkk0k\}$	7
$\{kk00, k0k00, kkkk00, k000k00, k00kk0k, k0k0kk0, kkkk0kk\}$	7
$\{k00k, kk0kk, k0kk0k, kkk0kkk, kk000, k0k000, kkkk000\}$	7
$\{kkk0k, k00kkk, kk0k00k, kkk0k0, k00kkk0, k0k00kk, kkk0k00\}$	7
$\{k0kkk, kkk00k, k00k0kk, k0kkk00, kk00k0k, k0kkk0, kkk00k0\}$	7
$\{k00k0, kk0kk0, k0kk0k0, kk0kkkk, kk0000, k0k0000, kkk000k\}$	7
$\{k0000k, kk000kk, k00k00, kk0kk00, kk0k0k, k0kkkkk, kk00000\}$	7
$\{kkkkkk, k00000k, k0000k0, k000kkk, k00k000, k0kk00k, kk0k0k0\}$	7
$n = 8$	
$\{0\}$	1

D_n-orbits	Length
$n = 9$	
$\{0\}$	1
$\{kk0kk0kk, k0kk0kk0k, kk0kk0kk0\}$	3
$\{kk, k0k, kkkk, k000k, kk00kk, k0k0k0k, kkkkkkkk, k0000000k,$ $k000000k0, k00000kkk, k0000k000, k000kk00k, k00k0k0k0, k0kkkkkkk,$ $kk0000000, k000000k, kk00000kk, k0000k00, kk000kk00, k00k0k0k,$ $kk0kkkkkk, kk000000, k0k000000, kkk00000k, k0000k0, kk000kk0,$ $k0k00k0k0, kkk0kkkkk, kk00000, k0k00000, kkkk00000, k0000k, kk000kk,$ $k0k00k0k, kkkk0kkkk, kk0000, k0k0000, kkkk0000, k000k0000, k00kk000k,$ $k0k0k00k0, kkkkk0kkk, kk000, k0k000, kkkk000, k000k000, kk00kk000,$ $k0k0k00k, kkkkkk0kk, kk00, k0k00, kkkk00, k000k00, kk00kk00, k0k0k0k00,$ $kkkkkkk0k, kk0, k0k0, kkkk0, k000k0, kk00kk0, k0k0k0k0, kkkkkkkk0\}$	63
$\{k00k, kk0kk, k0kk0k, kkk0kkk, k00kk00k, kk0k0k0kk, kkkkkk00,$ $k00000k00, k0000kk0k, k000k0kk0, k00kkk0kk, k0k00kk00, kkk0k0k0k,$ $kkkkkk0, k00000k0, kk0000kk0, k000k0kk, kk00kkk0k, k0k00kk0,$ $kkkk0k0k0, kkkkkk, k00000k, kk0000kk, k0k000k0k, kkk00kkk0, k0k00kk,$ $kkkk0k0k, k000kkkkk, k00k00000, k0kk0000k, kk0k000k0, kkk00kkk,$ $k00k0k00k, k0kkkk0k0, kk000kkkk, k00k0000, kk0kk0000, kk0k000k,$ $k0kkk00kk, kk00k0k00, k0kkkk0k, kkk000kkk, k00k000, kk0kk000,$ $k0kk0k000, kk0kkk00k, kk00k0k0, k0k0kkkk0, kkkk000kk, k00k00, kk0kk00,$ $k0kk0k00, kkk0kkk00, kk00k0k, k0k0kkkk, kkkkk000k, k00k0, kk0kk0,$ $k0kk0k0, kkk0kkk0, k00kk00k0, k0k0k0kkk, kkkkkk000\}$	63
$\{kkk0k, k00kkk, kk0k00k, k0kkk0kk, kkk00kk0k, k0k0kk0, kkkkk0k0,$ $k0000kkk0, k000k00kk, k00kk0k00, k0k0kkk0k, kkkk00kk0, k0k0kk,$ $kkkkk0k, k0000kkk, kk000k00k, k00kk0k0, kk0k0kkk0, kkkk00kk,$ $k000k0k0k, k00kkkk0, k0k0000kk, kkk000k00, k00kk0k, kk0k0kkk,$ $k0kkkk00k, kk000k0k0, k00kkkk, kk0k0000k, kkk000k0, k00k00kk0,$ $k0kk0k0kk, kk0kkk00, kk000k0k, k0k00kkkk, kkk0k0000, kkk000k,$ $k00k00kk, kk0kk0k0k, kk0kkkk0, k0kk000k0, kk0k00kkk, kkk0k000,$ $k00kkk000, k0k00k00k, kkk0kk0k0, kk0kkkk, k0kk000k, kkk0k00kk, kkk0k00,$ $k00kkk00, kk0k00k00, kkk0kk0k, k00kk0kkk, k0k0kk000, kkkk0k00k,$ $kkk0k0, k00kkk0, kk0k00k0, k0kkk0kk0, kk00kk0kk, k0k0kk00, kkkkk0k00\}$	63
$\{k0kkk, kkk00k, k00k0kk, kk0kkk0k, k0kk00kkk, kk0k0k000, kkkkk00k,$ $k0000k0kk, k000kkk00, k00k00k0k, k0kk0kkk0, kk0kk00kk, kk0k0k00,$ $k0kkkkk00, kk0000k0k, k000kkk0, kk00k00k0, k0kk0kkk, kkk0kk00k,$ $kk0k0k0, k0kkkkk0, kkk0000k0, k000kkk, kk00k00k, k0k0kk0kk,$ $kkkk0kk00, kk0k0k, k0kkkkk, kkk0000k, k00k000kk, k0kk00k00, kk0k0kk0k,$ $kkkk0kk0, k000kk0k0, k00k0kkkk, k0kkk0000, kk00k000k, k0kk00k0,$ $kkk0k0kk0, kkkk0kk, k000kk0k, kk00k0kkk, k0kkk000, kkk00k000, k0kk00k,$ $kkk0k0kk, k00kkkk0k, k0k000kk0, kkk00k0kk, k0kkk00, kkk00k00,$ $k00k0kk00, k0kkk0k0k, kk00kkkk0, k0k000kk, kkkk00k0k, k0kkk0,$ $kkk00k0, k00k0kk0, kk0kkk0k0, kk00kkkk, k0k0k000k, kkkkk00k0\}$	63

D_n-orbits	Length
$n=10$	
$\{0\}$	1
$\{kk000kk, k0k00k0k, kkkk0kkkk, k000kk000k, k00k0k00k0, k0kkkk0kkk, kk000kk000, k00k0k00k, kk0kkkk0kk, kk000kk00, k0k00k0k00, kkk0kkkk0k, kk000kk0, k0k00k0k0, kkkk0kkkk0\}$	15
$\{k0k, kkkk, k000k, kk00kk, k0k0k0k, kkkkkkkk, k0000000k, kk000000kk, k00000k00, kk0000kk00, k000k0k0k, kk00kkkkk, k0k000000, kkkk000000, k00000k, kk0000kk, k0k000k0k, kkkk00kkkk, k0k0000, kkkk0000, k000k0000, kk00kk0000, k0k0k000k, kkkkk00kk, k0k00, kkkk00, k000k00, kk00kk00, k0k0k0k00, kkkkkkk00\}$	30
$\{k0k0, kkkk0, k000k0, kk00kk0, k0k0k0k0, kkkkkkkk0, k0000000k0, k000000kkk, k00000k000, k0000kk00k, k000k0k0k0, k00kkkkkkk, k0k0000000, kkk000000k, k00000k0, kk0000kk0, k0k000k0k0, kkk00kkkkk, k0k00000, kkkk00000, k000k00000, k00kk0000k, k0k0k000k0, kkkkk00kkk, k0k000, kkkk000, k000k000, kk00kk000, k0k0k0k000, kkkkkkk00k\}$	30
$\{kk0kk, k0kk0k, kkk0kkk, k00kk00k, kk0k0k0kk, k0kkkkkk0k, kk00000kk0, k0000k0kk, kk000kkk0k, k00k00kk0, kk0kk0k0k0, kk0kkkkkk, k0kk00000k, kk0k0000k0, kkk000kkk, k00k00k00k, k0kk0kk0k0, kk0kk0kkkk, kk0kk0000, k0kk0k0000, kk0kkk000k, kk00k00k0, k0k0kk0kk0, kkkk0kk0kk, kk0kk00, k0kk0k00, kkk0kkk00, k00kk00k00, k0k0k0kk0k, kkkkkk0kk0\}$	30
$\{k00kkk, kk0k00k, k0kkk0kk, kkk00kk0k, k00k0k0kkk, k0kkkkk000, kk0000k00k, k000kk0k0, kk00k0kkk0, k0kkk00kk, kkk00k0k0k, k0kkkkk0, kkk0000k0, k00k000kk0, k0kk00k0kk, kk0k0kkk00, kkkk00k0k, k000k0kkkk, k00kkk0000, k0k00k000k, kkk0kk00k0, kk0k0kkk, k0kkkk00k, kkk000k0kk, k00kkk00, kk0k00k00, k0kkk0kk00, kk00kk0k0k, k0k0kkkk0, kkkkk000k0\}$	30
$\{kkk00k, k00k0kk, kk0kkk0k, k0kk00kkk, kkk0k0k00k, kkkkk0k0, k0000kkk0, kk000k00k0, k00kk0kkk, kk0k0kk00k, kkkk0k0k0, k000kkkkk0, k00k0000kk, k0kk000k00, kk0k00kk0k, kkk0k0kk0, k00kkkk0k0, k0k000kkkk, kkk00k0000, k0kk000k, kkk0k00kk, k00kkk0k0k, k0k00kkkk0, kkk0k000kk, kkk00k00, k00k0kk00, kk0kkk0k00, kk00kkk0k, k0k0k00kkk, kkkkk0k000\}$	30
$\{kk0kk0, k0kk0k0, kkk0kkk0, k00kk00k0, kk0k0k0kk0, kkkkkk0kk, k00000kk0k, k0000k0kk0, k000kkk0kk, k00k00kk00, k0kk0k0k0k, kk0kkkkkk0, kk00000kk, k0k0000k0k, kkk000kkk0, k00k00kk, kk0kk0k0k, k0kk0kkkkk, kk0kk00000, kk0k0000k, k0kkk000kk, kk00k00k00, k0kk0kk0k, kkk0kk0kkk, kk0kk000, k0kk0k000, kkk0kkk000, kk00k00k, k0k0kk0kk, kkkkk0kk0k\}$	30
$\{k00kkk0, kk0k00k0, k0kkk0kk0, kkk00kk0k0, k0k0kkkk, kkkkk000k, k0000k00kk, k000kk0k00, k00k0kkk0k, k0kkk00kk0, kk00k0k0kk, k0kkkkk00, kkk0000k00, k000kk0k, kk00k0kkk, k0k0kkk00k, kkkk00k0k0, k0kkkkk, kkk0000k, k00k000kk, kk0kk00k0k, kk0k0kkk0, k0kkkk00k0, kk000k0kkk, k00kkk000, kk0k00k000, kkk0kk00k, k00kk0k0kk, k0k0kkk00, kkkk000k0k\}$	30
$\{kkk00k0, k00k0kk0, kk0kkk0k0, k0kk00kkk0, kk0k0k00kk, kkkkk0k00, k0000kkk00, k000k00k0k, k00kk0kkk0, k0k0kk00kk, kkkk0k0k00, kkkkk0k, k0000kkk, kk000k00k, k0k00kk0kk, kkk0k0kk00, kkkk0k0k, k000kkkkk, kk00k0000k, k0kk000k0, kkk0k00kk0, kkk0k0kk, k00kkkk0k, kk0k000kkk, kkk00k000, k00k0kk000, k0kkk0k00k, kk00kkk0k0, k0k00kkkk, kkkk0k000k\}$	30

As for $n > 10$ the orbits become too large to be written explicitly, we only give their number and cardinality.

Fact 4.3. *All nontrivial D_n-orbits for $11 \leqslant n \leqslant 17$ are the following:*

1. $n = 11$*: twenty seven 341-element orbits,*
2. $n = 12$*: nine 3-element orbits and 198 orbits of cardinality 12 each,*
3. $n = 13$*: forty five 819-element orbits,*
4. $n = 14$*: eighty one 7-element orbits and 2592 orbits of cardinality 14 each,*
5. $n = 15$*: nine 3-element orbits, twenty seven 5-elements orbits and 9819 orbits of cardinality 15 each,*
6. $n = 16$*: no nontrivial orbits,*
7. $n = 17$*: twenty seven 85-element orbits and 2304 orbits of cardinality 255 each.* □

Remark 4.4. Consider a number containing exactly two distinct digits – 0 and some nonzero A. Since digits appear in bunches, we shall use the notation introduced in Section 2.2. So there are 4 possible forms of such numbers, that is

$$D_n(0_{k_1}A_{s_1}\ldots 0_{k_t}A_{s_t}) = (0_{k_1-1}A_1)(0_{s_1-1}A_1)\ldots(0_{k_t-1}A_1)(0_{s_t-1}A_1)$$

$$D_n(A_{s_1}0_{k_1}\ldots A_{s_t}0_{k_t}) = (0_{s_1-1}A_1)(0_{k_1-1}A_1)\ldots(0_{k_t-1}A_1)(0_{s_t-1}A_1)$$

$$D_n(0_{k_1}A_{s_1}\ldots 0_{k_t}A_{s_t}0_{k_{t+1}}) = (0_{k_1-1}A_1)(0_{s_1-1}A_1)\ldots(0_{k_t-1}A_1)(0_{s_t-1}A_1)0_{k_{t+1}}$$

$$D_n(A_{s_1}0_{k_1}\ldots A_{s_t}0_{k_t}A_{s_{t+1}}) = (0_{s_1-1}A_1)(0_{k_1-1}A_1)\ldots(0_{k_t-1}A_1)(0_{s_t-1}A_1)0_{s_{t+1}}$$

whence we always get an even number of appearances of A. That leads to a possibility that elements of this form appear in D_n-orbits. Note however that $D_n(A_n) = 0$ and $D_n(0_n) = 0$, so it can be untrue for even n (and it is actually for powers of 2, as follows from the above theorem). If we assume now that n is odd, then we always get at least one 0 whence at least one A, so we again obtain a number which is in one of the 4 forms.

Examination of numerical results for D_n-orbits leads to the following

Conjecture 3. Each element of a nontrivial D_n-orbit for $n \in \mathbb{N}$ is a number containing only two distinct digits – a nonzero digit $k \in \{1, 2, \ldots, 9\}$, which always has an even number of appearances, and 0.

Remark 4.5. We have considered one more example of general Kaprekar's transformation which is a generalization of D_n, namely $d_{id,\pi_{n,r}}$, where for every $r, n \in \mathbb{N}$, $r < n$ we have

$$\pi_{n,r}(k) = \begin{cases} k + r, & k = 1, 2, \ldots, n - r, \\ k - n + r, & k = n - r + 1, \ldots, n, \end{cases}$$

It turned out that for $r = 2$ and odd n orbits of $d_{id,\pi_{n,2}}$ and D_n form the same sets, but elements in orbits appear in a different order (i.e. arise from different iterations). For instance, nontrivial orbits for $d_{id,\pi_{3,2}}$ are

$$\{k0k,\ 0kk,\ kk0\}, \quad A = 1, 2, \ldots, 9,$$

(compare to D_3-orbits given in the table above) and nontrivial orbits for $d_{id,\pi_{5,2}}$ are

$$\{kkkk,\ k00k0,\ kk000,\ kk0kk,\ k0k00,\ kk0,\ kkkk0,\ k0k,\ k000k,\ k0kkk,\ k00k,\ kk00,\ kkk0k,\ k0k0,\ kk\},\quad k=1,2,\dots,9$$

(compare to D_5-orbits given in the table above).

5. Further generalizations Kaprekar's transformations

All transformations we have discussed so far are of the form

$$S_{f,g}(\alpha) := \sum_{k=1}^{n} |s_{f(k)} - s_{g(k)}| 10^{n-k},$$

where f, g are some permutations on $\{1, \dots, n\}$ and $s_1, \dots, s_n$ is a sequence of digits of α, either consecutive or nondecreasing. So the next step in generalizing could be considering the case when f, g are **any** functions. We only mention it as a remark as considering these transformations in details would exceed the scope of this paper.

One more idea is to introduce Kaprekar-style transformations in some other algebraic structures, like groups or rings. As the example we propose two transformations on symmetric groups S_n, $n \geqslant 2$. Recall that each permutation can be uniquely (up to the order) represented as a product of pairwise disjoint cycles. So let a permutation $\pi \in S_n$ be written as a product of pairwise disjoint cycles such that their lengths decreases and cycles of length 1 are omitted in this notation. Now, let max_π (resp. min_π) be a permutation nontrivial cycles of which have the same lengths as those in π and elements in these cycles are consecutive numbers in decreasing (resp. increasing) order starting with n (resp. 1). For example, if $\pi = (2,4,6,8)(1,3,5) \in S_9$ then $max_\pi = (9,8,7,6)(5,4,3)$ and $min_\pi = (1,2,3,4)(5,6,7)$. Next, let $max_{supp(\pi)}$, $min_{supp(\pi)}$ be permutations defined analogously as max_π, min_π, respectively, but on the support of π only, that is on $supp(\pi) := \{k \in \mathbb{N} \colon k \leqslant n \text{ and } \pi(k) \neq k\}$. For example if $\pi = (2,4,6,8)(1,3,5) \in S_9$ then $max_{supp(\pi)} = (8,6,5,4)(3,2,1)$, $min_{supp(\pi)} = (1,2,3,4)(5,6,8)$. Note that this idea corresponds to ordering digits of numbers in case of classical Kaprekar's transformations. Now we define two Kaprekar-style transformations, namely for every permutation $\pi \in S_n$ we have

$$\mathcal{F}_n(\pi) = max_\pi \circ min_\pi, \qquad \mathcal{R}_n(\pi) = max_{supp(\pi)} \circ min_{supp(\pi)} \tag{10}$$

As the example consider $\pi = (1342)(657) \in S_7$. Then

$$\begin{aligned}
\mathcal{F}_7(\pi) &= (7654)(321)(1234)(567) = (374),\\
\mathcal{F}_7^2(\pi) &= (765)(123),\\
\mathcal{F}_7^3(\pi) &= (765)(432)(123)(456) = (14763),\\
\mathcal{F}_7^4(\pi) &= (76543)(12345) = (12765),
\end{aligned}$$

whence

$$\mathcal{F}_7^n(\pi) = (12765), \quad n \geqslant 4$$

and

$$\begin{aligned}\mathcal{R}_7(\pi) &= (7654)(321)(1234)(567) = (374),\\ \mathcal{R}_7^2(\pi) &= (743)(347) = \mathrm{id}.\end{aligned}$$

Basing on the numerical calculations we performed we suspect the following is true:

Conjecture 4. For every natural n and each permutation $\pi \in S_n$ there exists a natural number $k = k(\pi)$ such that $\mathcal{R}_n^k(\pi) = \mathrm{id}$.

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